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INTEGRAL CHARACTERISTICS OF RIGID INCLUSIONS AND CAVITIES IN THE TWO-DIMENSIONAL THEORY OF ELASTICITY†

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Representations of the components of the elastic-polarization matrices and the Wiener elastic capacity are obtained in terms of the coefficients of the Kolosov–Muskhelishvili complex potentials and the coefficients of the conformal representation, which define the geometry of an infinite elastic solid. A new integral characteristic of a rigid inclusion—the Roben matrix, whose components are dimensionless, is proposed for use in applied problems. Examples of calculations, which correct formulae published previously elsewhere, are given. © 1998 Elsevier Science Ltd. All rights reserved.

1. THE POLARIZATION MATRICES AND WIENER CAPACITIES

An asymptotic analysis of problems of the interaction between cracks and inclusions in an elastic solid leads to the need to investigate the integral characteristics of these defects, such as the elastic-polarization matrices and the Wiener capacities. The latter are extensions of the corresponding classical objects from the theory of harmonic functions [1].‡ In two-dimensional problems the Kolosov–Muskhelishvili methods enable one to calculate these quantities effectively. Nevertheless, a number of incorrect results have been published ([3], etc.), which is also our purpose to correct here.

Suppose G is a region in the $\mathbb{R}^2$ plane, bounded by a simple closed piecewise-continuous contour Γ. We will denote the closed region $G \cup \Gamma$ by $\overline{G}$ and we will put $\Omega = \mathbb{R}^2 \backslash \overline{G}$.

Consider the equations of the first fundamental problem of the two-dimensional theory of elasticity

$$\mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} = (x_1, x_2) \in \Omega \tag{1.1}$$

$$\begin{gathered}\boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x}) = \mathbf{p}(\mathbf{x}), \quad \mathbf{x} \in \Gamma \\ \boldsymbol{\sigma}^{(n)} = (\sigma_{11}n_1 + \sigma_{12}n_2, \sigma_{21}n_1 + \sigma_{22}n_2)^t \\ \sigma_{kl} = \lambda\delta_{kl}(\partial_1 u_1 + \partial_2 u_2) + \mu(\partial_l u_k + \partial_k u_l), \quad \partial_k = \partial/\partial x_k\end{gathered} \tag{1.2}$$

Here λ, μ are the Lamé constants, $\mathbf{u} = (u_1, u_2)^t$ is the column vector of displacements, t is the sign of transposition, $\boldsymbol{\sigma}^{(n)}$ is the vector of the stresses on an area with unit normal $\mathbf{n} = (u_1, u_2)^t$ to Γ (outward with respect to Ω), σ_{kl} are the components of the stress tensor, δ_{kl} is the Kronecker delta, and $\mathbf{p} = (p_1, p_2)^t$ is a specified vector of the external load.

To obtain the correct boundary-value problem, it is necessary to add to relations (1.1) and (1.2) the condition which characterizes the behaviour of $\mathbf{u}(\mathbf{x})$ as $|\mathbf{x}| = (x_1^2 + x_2^2)^{1/2} \to \infty$.

We know that a unique solution of problem (1.1), (1.2) (apart from a rigid displacement), whose components have no more than a logarithmic singularity at infinity, has the asymptotic representation

$$\begin{gathered}\mathbf{u}(\mathbf{x}) = -T(\mathbf{x})\mathbf{F} - \frac{M}{2}\left\{\frac{\partial \mathbf{T}^{(2)}}{\partial x_1}(\mathbf{x}) - \frac{\partial \mathbf{T}^{(1)}}{\partial x_2}(\mathbf{x})\right\} + \sum_{j=1}^{3} c_j[\mathbf{V}^{(j)}(\nabla_x)T(\mathbf{x})]^t + O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}| \mapsto \infty \\ \nabla_x = (\partial/\partial x_1, \partial/\partial x_2), \quad \varkappa = (\lambda+3\mu)(\lambda+\mu)^{-1}\end{gathered} \tag{1.3}$$

Here $\mathbf{F} = (F_1, F_2)^t$ and M are the principal vector and principal moment (with respect to the origin

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‡See also BABICH, V. M., ZORIN, I. S., IVANOV, M. A., MOVCHAN, A. B. and NAZAROV, S. A., *Integral characteristics in problems of the theory of elasticity*. Preprint No. R-6-89. Izd.-vo Leningrad. Optiko-Mekh. Inst., Leningrad, 1989.

of coordinates) of the load $\mathbf{p}$, ∇_x is the Hamilton operator, the column vectors $\mathbf{V}^{(j)}(\mathbf{x})$ are such that $\mathbf{V}^{(1)}(\mathbf{x}) = (x_1, 0)$, $\mathbf{V}^{(2)}(\mathbf{x}) = (O, x_2)$, $\mathbf{V}^{(3)}(\mathbf{x}) = (x_2, x_1)$, and $\mathbf{T}^{(k)}$ are the columns of the matrix

$$T(\mathbf{x}) = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\begin{Vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22}\end{Vmatrix}, \quad \begin{matrix} t_{kk} = -2\varkappa \ln|\mathbf{x}| + 2x_k^2 |\mathbf{x}|^{-2} \\ t_{12} = t_{21} = 2x_1 x_2 |\mathbf{x}|^{-2}\end{matrix} \tag{1.4}$$

Matrix (1.4) defines the so-called [4] influence tensor in an unbounded elastic two-dimensional medium. Here $\mathbf{T}^{(k)}$ is the displacement vector of points of the elastic half-plane, loaded with a unit point force applied at the origin of coordinates and directed along the Ox_k ($k = 1, 2$) axis. The vectors $\mathbf{V}^{(j)}(\mathbf{x})$ ($j = 1, 2, 3$), together with the rotation $\mathbf{V}^{(4)}(\mathbf{x}) = (-x_2, x_1)$ form a basis in the space of homogeneous vector polynomials of the first degree. Note that the expression in the braces on the right in (1.3) can be written as $[\mathbf{V}^{(4)}(\nabla_x)\, T(\mathbf{x})]^t$. Using the method proposed previously [5] the constants c_j can be expressed by the formula

$$c_j = \int_\Gamma \mathbf{U}^{(j)}(\mathbf{x})^t \mathbf{p}(\mathbf{x}) ds_x \tag{1.5}$$

where $\mathbf{U}^{(j)}$ are special solutions of homogeneous problem (1.1), (1.2), characterized at infinity by the asymptotic form

$$\mathbf{U}^{(j)}(\mathbf{x}) = \mathbf{V}^{(j)}(\mathbf{x})^t + \sum_{m=1}^{3} P_{jm}[\mathbf{V}^{(m)}(\nabla_x)T(\mathbf{x})]^t + O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}| \mapsto \infty \tag{1.6}$$

The symmetric matrix $\| P_{jm} \|_{j,m=1}^3$ is called the elastic-polarization matrix [6]. Formula (1.5) enables integral representations to be derived for the coefficients P_{jm}. This justifies the term "integral" used in the title of this paper.

When investigating the behaviour at infinity of the solutions of the second fundamental problem of the two-dimensional theory of elasticity, in which Eq. (1.1) and the boundary condition

$$\mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x} \in \Gamma \tag{1.7}$$

occur, where $\mathbf{g} = (g_1, g_2)^t$ is a specified displacement vector, special solutions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ of homogeneous problem (1.1), (1.7), defined at infinity by the asymptotic form

$$\mathbf{S}^{(k)}(\mathbf{x}) = \mathbf{T}^{(k)}(\mathbf{x}) + \mathbf{D}^{(k)} + O(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \mapsto \infty \tag{1.8}$$

are used.

The symmetric matrix $\| D_l^{(k)} \|_{l,k=1}^2$ with columns $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ is called the Wiener elastic capacity matrix [7].

The properties of these vectors, and also examples of their use were given in [6, 7].

2. AN EXPRESSION FOR THE COMPONENTS OF THE POLARIZATION MATRIX AND THE WIENER CAPACITY IN TERMS OF THE COEFFICIENTS OF THE COMPLEX POTENTIALS

We will introduce the complex variable $z = x_1 + ix_2$ and recall Kolosov's formula for the displacements

$$2\mu(u_1 + iu_2)(z) = \varkappa\varphi_1(z) - z\overline{\varphi_1'(z)} - \overline{\psi_1(z)} \tag{2.1}$$

where the prime denotes differentiation with respect to z and the bar denotes the operation of complex conjugation. Assuming that the components of the stress tensor are bounded over the whole region Ω, the following expansions [8, Section 36] hold for the complex potentials for sufficiently large $|z|$

$$\varphi_1(z) = -f\ln z + a_1 z + a_0 + \frac{a_{-1}}{z} + \ldots, \quad \psi_1(z) = \varkappa\bar{f}\ln z + b_1 z + b_0 + \frac{b_{-1}}{z} + \ldots \tag{2.2}$$

$$f = \frac{F_1 + iF_2}{2\pi(1+\varkappa)}$$

In the problem of calculating the components of the polarization matrix (see (1.6)) we must put $F_1 = F_2 = 0$ and also $\varkappa a_0 - \bar{b}_0 = 0$. Correspondingly, substituting (2.2) into (2.1) we obtain

$$2\mu(U_1^{(j)} + iU_2^{(j)})(z) = (\varkappa a_1^{(j)} - \bar{a}_1^{(j)})z - \bar{b}_1^{(j)}z + \frac{\varkappa a_{-1}^{(j)}\bar{z} - \bar{b}_{-1}^{(j)}z}{|z|^2} + \frac{\bar{a}_{-1}^{(j)}z^3}{|z|^4} + \dots \tag{2.3}$$

We will compare (1.6) and (2.3). The term $\mathbf{V}^{(j)}(\mathbf{x})^t$ defines the coefficients $a_1^{(j)}$ and $b_1^{(j)}$

$$a_1^{(j)} = \frac{1}{2}(\lambda + \mu), \quad b_1^{(j)} = (-1)^j \mu, \quad j = 1,2; \quad a_1^{(3)} = 0, \; b_1^{(3)} = i2\mu \tag{2.4}$$

Comparing terms of the order of $|z|^{-1}$ we have

$$P_{jk} = \frac{\pi(\lambda + 2\mu)}{\mu} b_{-1}^{(j)} + (-1)^k \frac{2\pi(\lambda + 2\mu)}{\lambda + \mu} \operatorname{Re} a_{-1}^{(j)}, \quad k = 1,2 \tag{2.5}$$

$$P_{j3} = -\frac{2\pi(\lambda + 2\mu)}{\lambda + \mu} \operatorname{Im} a_{-1}^{(j)}, \quad j = 1,2,3$$

Note that the coefficients $b_{-1}^{(j)}$ must necessarily be real, since its imaginary part, apart from a factor (see, for example, [8, Section 56a, Paragraph 5]) is equal to the moment of the load $\mathbf{p}^{(j)}(\mathbf{x}) = -\sigma^{(n)}(\mathbf{V}^{(j)}(\mathbf{x})^t)$, which is statically self-balanced.

In the problem of calculating the components of the Wiener capacity matrix (see (1.8)) we must take $a_1 = b_1 = 0$ in (2.2). Substituting (2.2) into (2.1) we obtain the relation

$$2\mu(S_1^{(k)} + iS_2^{(k)})(z) = -\frac{\varkappa(F_1^{(k)} + iF_2^{(k)})}{\pi(1+\varkappa)} \ln|z| + \frac{F_1^{(k)} - iF_2^{(k)}}{2\pi(1+\varkappa)} \frac{z^2}{|z|^2} + \varkappa a_0^{(k)} - \bar{b}_0^{(k)} + \dots;$$

$$F_l^{(k)} = \delta_{lk}, \quad l = 1,2 \tag{2.6}$$

Separating the sum $T_1^{(k)} + iT_2^{(k)}$ on the right in (2.6) (see (1.4)), we compare it with (1.8). We obtain

$$2\mu(D_1^{(k)} + iD_2^{(k)}) = -\frac{\delta_{1k} + i\delta_{2k}}{2\pi(1+\varkappa)} + \varkappa a_0^{(k)} - \bar{b}_0^{(k)}, \quad k = 1,2 \tag{2.7}$$

Note that one of the complex constants $a_0^{(k)}, b_0^{(k)}$ can be equated to zero, thereby reducing a number of unknowns on both sides of (2.7) to equality.

3. THE USE OF A CONFORMAL TRANSFORMATION

Suppose the region Ω is in the form of the exterior $|\zeta| > 1$ of the unit circle with the conformal representation

$$z = \omega(\zeta) = c\zeta + c_1\zeta^{-1} + c_2\zeta^{-2} + \dots \tag{3.1}$$

Substituting the expression from the right-hand side of (3.1) into (2.2) instead of z we obtain

$$\varphi(\zeta) = \varphi_1[\omega(\zeta)] = -f \ln\zeta + A_1\zeta + A_0 + A_{-1}\zeta^{-1} + \dots \tag{3.2}$$

$$\psi(\zeta) = \psi_1[\omega(\zeta)] = \varkappa \bar{f} \ln\zeta + B_1\zeta + B_0 + B_{-1}\zeta^{-1} + \dots$$

$$A_1 = ca_1, \qquad B_1 = cb_1 \tag{3.3}$$

Here the following relations occur between the coefficients of the potentials $\varphi(\zeta)$ and $\varphi_1(z)$, $\psi(\zeta)$ and $\psi_1(z)$

$$a_0 = f\ln c + A_0,\quad b_0 = -\varkappa\bar{f}\ln c + B_0;\quad a_{-1} = cA_{-1} - c_1A_1,\quad b_{-1} = cB_{-1} - c_1B_1 \tag{3.4}$$

Hence, instead of (2.5) we have the following expressions for the components of the polarization matrix

$$P_{jk} = \frac{\pi(\lambda+2\mu)}{\mu}(cB_{-1}^{(j)} - c_1B_1^{(j)}) + (-1)^k\frac{2\pi(\lambda+2\mu)}{\lambda+\mu}\operatorname{Re}(cA_{-1}^{(j)} - c_1A_1^{(j)}),\quad k=1,2 \tag{3.5}$$

$$P_{j3} = \frac{2\pi(\lambda+2\mu)}{\lambda+\mu}\operatorname{Im}(cA_{-1}^{(j)} - c_1A_1^{(j)}),\ j=1,2,3$$

The constants $A_1^{(j)}$, $B_1^{(j)}$, by (2.4) and (3.3), take the values

$$A_1^{(j)} = \frac{1}{2}(\lambda+\mu)c,\quad B_1^{(j)} = (-1)^j\mu c,\quad j=1,2;\quad A_1^{(3)} = 0,\quad B_1^{(3)} = i2\mu c \tag{3.6}$$

Correspondingly, we can derive the following formula for the components of the Wiener capacity matrix, which generalizes (2.7)

$$2\mu(D_1^{(k)} + iD_2^{(k)}) = -\frac{\delta_{1k}+i\delta_{2k}}{2\pi(1+\varkappa)}(1-2\varkappa\ln|c|) + \varkappa A_0^{(k)} - \bar{B}_0^{(k)},\quad k=1,2 \tag{3.7}$$

Suppose now that Ω is the transform of the interior $|\zeta| < 1$ of the unit circle with a conformal representation corresponding to replacing ζ by ζ^{-1} on the right-hand side of (3.1). As before, we express the components of the polarization matrix and the Wiener capacity in terms of the potential coefficients and the conformal representation. For example, for the components of the polarization matrix representations hold which differ from (3.5) by the replacement of $A_{-1}^{(j)}$, $A_1^{(j)}$ and $B_{-1}^{(j)}$, $B_1^{(j)}$ by $\alpha_1^{(j)}$, $\alpha_{-1}^{(j)}$ and $\beta_1^{(j)}$, $\beta_{-1}^{(j)}$ respectively, where $\alpha_1^{(j)}$ and $\beta_1^{(j)}$ are the coefficients of ζ^{-1} in the expansions of the potentials $\alpha_1^{(j)}$ and $\beta_1^{(j)}$, i.e.

$$\alpha_{-1}^{(j)} = \frac{1}{2}(\lambda+\mu)c,\quad \beta_{-1}^{(j)} = (-1)^j\mu c,\quad j=1,2;\quad \alpha_{-1}^{(3)} = 0,\quad \beta_{-1}^{(3)} = i2\mu c$$

These formulae enable one to use ready results, obtained previously [10, etc.].

4. EXAMPLES

Suppose the region Ω, which does not contain the origin of coordinates, is the transform of the exterior of the unit circle in the conformal representation

$$z = \omega(\zeta) = c\zeta + c_1\zeta^{-1} + c_2\zeta^{-2} \tag{4.1}$$

In the problem of constructing the matrix of the Wiener elastic capacity, the potentials (3.2) can be written as

$$\varphi^{(k)}(\zeta) = -\frac{\delta_{1k}+i\delta_{2k}}{2\pi(1+\varkappa)}\ln\zeta + \varphi_0^{(k)}(\zeta),\quad \psi^{(k)}(\zeta) = \frac{\varkappa(\delta_{1k}-i\delta_{2k})}{2\pi(1+\varkappa)}\ln\zeta + \psi_0^{(k)}(\zeta) \tag{4.2}$$

where $\varphi_0^{(k)}$ and $\psi_0^{(k)}$ are holomorphic functions outside the unit circle, including an infinitely distant point, i.e.

$$\varphi_0^{(k)}(\zeta) = A_0^{(k)} + A_{-1}^{(k)}\zeta^{-1} + \ldots,\quad \psi_0^{(k)}(\zeta) = B_0^{(k)} + B_{-1}^{(k)}\zeta^{-1} + \ldots,\quad k=1,2 \tag{4.3}$$

Using the fact that one of the constants $A_0^{(k)}$, $B_0^{(k)}$ can be assumed to be equal to zero, we put $A_0^{(k)} = 0$ ($k = 1, 2$). The homogeneous boundary condition (1.7) for the complex potentials (see [8, Section 51]), taking (4.2) into account, has the form

$$\varkappa\varphi_0^{(k)}(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0^{(k)\prime}(\sigma)} - \overline{\psi_0^{(k)}(\sigma)} = -\frac{\delta_{1k}-i\delta_{2k}}{2\pi(1+\varkappa)}\sigma\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}},\quad |\sigma|=1 \tag{4.4}$$

Here, by (4.1)

$$\frac{\omega(\sigma)}{\omega'(\sigma)}=\frac{c\sigma^3+c_1\sigma+c_2}{\sigma^2(\bar{c}-\bar{c}_1\sigma^2-2\bar{c}_2\sigma^3)}$$

The following functional Muskhelishvili functional equation for determining the function $\varphi_0^{(k)}$ corresponds to boundary condition (4.4)

$$-\varkappa\varphi_0^{(k)}(\zeta)-\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0^{(k)\prime}(\sigma)}\frac{d\sigma}{\sigma-\zeta}=-\frac{\delta_{1k}-i\delta_{2k}}{2\pi(1+\varkappa)}\frac{1}{2\pi i}\int_\gamma \sigma\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{d\sigma}{\sigma-\zeta} \tag{4.5}$$

where ζ is an arbitrary point outside the unit circle and γ is the circle $|\sigma|=1$. Since the expression $[\omega(\sigma)/\overline{\omega'(\sigma)}]\overline{\varphi_0^{(k)\prime}(\sigma)}$ is the boundary value of the function

$$\frac{c\zeta^3+c_1\zeta+c_2}{\zeta^2(\bar{c}-\bar{c}_1\zeta^2-2\bar{c}_2\zeta^3)}\overline{\varphi_0^{(k)\prime}}\left(\frac{1}{\zeta}\right)$$

which is holomorphic inside the unit circle, the integral on the left-hand side of (4.5) is equal to zero. On the other hand, the expression under the integral on the right-hand side in (4.5) is the boundary value of the function

$$\zeta\frac{\omega(\zeta)}{\overline{\omega}'\left(\dfrac{1}{\zeta}\right)}=\frac{c\zeta^3+c_1\zeta+c_2}{\zeta(\bar{c}-\bar{c}_1\zeta^2-2\bar{c}_2\zeta^3)}=\frac{c_2}{\bar{c}}\frac{1}{\zeta}+\frac{c_1}{\bar{c}}+\frac{c_2\bar{c}_1}{\bar{c}^2}\zeta+\ldots \tag{4.6}$$

which is holomorphic inside the unit circle, with the exception of the origin of coordinates, where it has a pole of the first order with principal part $c_2\bar{c}^{-1}2\zeta^{-1}$. Consequently, by (4) from [8, Section 80], we obtain

$$\varphi_0^{(k)}(\zeta)=-\frac{\delta_{1k}-i\delta_{2k}}{2\pi\varkappa(1+\varkappa)}\frac{c_2}{\bar{c}}\frac{1}{\zeta} \tag{4.7}$$

We will now consider boundary condition (4.4). We substitute expression (4.7) into it and take into account expansion (4.6), which is applicable when $|\zeta|=1$. We arrive at a relation in which, by comparing the coefficients of σ^0, we obtain $B_0^{(k)}$. Finally, formula (3.7) gives

$$2\mu(D_1^{(k)}+iD_2^{(k)})=-\frac{\delta_{1k}+i\delta_{2k}}{2\pi(1+\varkappa)}\left(1-2\varkappa\ln|c|-\frac{1}{\varkappa}\frac{|c_2|^2}{|c|^2}\right)-\frac{\delta_{1k}-i\delta_{2k}}{2\pi(1+\varkappa)}\frac{cc_1}{|c|}$$

In expanded form, for the case (4.1), the matrix of the Wiener capacity looks like (compare with [2], formula (6), where, in particular, there are no terms with the factor $|c_2|^2$)

$$4\pi\mu(1+\varkappa)\begin{Vmatrix} D_1^{(1)} & D_1^{(2)} \\ D_2^{(1)} & D_2^{(2)} \end{Vmatrix}=2\varkappa\ln|c|E-\begin{Vmatrix} R_1^{(1)} & R_1^{(2)} \\ R_2^{(1)} & R_2^{(2)} \end{Vmatrix} \tag{4.8}$$

$$R_k^{(k)}=1+(-1)^{k+1}\operatorname{Re}\left(\frac{cc_1}{|c|^2}\right)-\frac{1}{\varkappa}\frac{|c_2|^2}{|c|^2},\quad R_2^{(1)}=R_1^{(2)}=\operatorname{Im}\left(\frac{cc_1}{|c|^2}\right) \tag{4.9}$$

where E is the identity matrix.

In the problem of calculating the components of the elastic polarization matrix, we can represent the potentials (3.2) as follows:

$$\varphi^{(j)}(\zeta)=A_1^{(j)}\zeta+\varphi_0^{(j)}(\zeta),\quad \psi^{(j)}(\zeta)=B_1^{(j)}\zeta+\psi_0^{(j)}(\zeta)$$

where $\varphi_0^{(j)}$ and $\psi_0^{(j)}$ are functions, holomorphic outside the unit circle, including an infinitely distant point, possessing expansions of the form (4.3). In view of the asymptotic form (1.6) the constants $A_0^{(j)}$ and $B_0^{(j)}$ must be related by the expression $\varkappa A_0^{(j)}-\bar{B}_0^{(j)}=0$ ($j=1,2,3$). In the same way as above, we obtain (compare with [3], where, in particular, there are no terms with the factor $|c_2|^2$)

$$A_{-1}^{(j)}=-\bar{A}_1^{(j)}\frac{c_1}{\bar{c}}-\bar{B}_1^{(j)} \tag{4.10}$$

$$B_{-1}^{(j)} = -A_1^{(j)}\left(\frac{|c|^2}{c^2}+\frac{|c_1|^2}{c^2}+\frac{2|c_2|^2}{c^2}\right)-\overline{A}_1^{(j)}\left(1+\frac{|c_1|^2}{|c|^2}+\frac{2|c_2|^2}{|c|^2}\right)-\overline{B}_1^{(j)}\frac{\overline{c}_1}{c}$$

Formulae (4.10), (3.5) and (3.6) solve the problem of determining the components of the elastic polarization matrix in case (4.1).

5. THE ROBEN MATRIX

The quantity $|c|$, where c is the coefficient in expansion (3.1), is called (see [1, Section 1.3]) the external conformal radius of the closed region $\overline{G}$, and has dimension of length. If r is the radius of the largest circle which is contained in $\overline{G}$ and R is the radius of the smaller circle which is contained in $\overline{G}$, we have the limits: $r \leqslant |c| \leqslant R$ (see, for example, [10, Section 4, problem No. 123]).

In applied investigations it is preferable to write the asymptotic formula (1.8) in the form

$$\mathbf{S}^{(k)}(\mathbf{x}) = \mathbf{T}^{(k)}\left(\frac{\mathbf{x}}{|c|}\right)-\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\mathbf{R}^{(k)}+O\left(\frac{1}{|\mathbf{x}|}\right), \quad |\mathbf{x}|\to\infty$$

where the logarithm of the dimensionless quantity is calculated.

The symmetric matrix $\| R_l^{(k)} \|_{l,k=1}^2$ with columns $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$ is called the Roben matrix, by analogy with the Roben constant [11, Section 3, Chapter 4].

The mapping $z = \omega(\zeta) = |c|(\zeta + m\zeta^{-1})$ transfers the exterior of the unit circle into the region Ω, the boundary of which is an ellipse with semiaxes $|c|(1 + m)$ and $|c|(1 - m)$. Using the explicit solution [8, Section 83a], we obtain $\mathbf{R} = \operatorname{diag}\{1 + m, 1 - m\}$. This matrix uses the property of positive definiteness when $|m| = 1$, corresponding to the degeneration of the ellipse into a segment.

If the geometry of the region Ω is specified by the conformal representation (4.1), the elements of the Roben matrix are defined by (4.9). We will show that when $c_2 \neq 0$ the Roben matrix is positive definite (this cannot be proved for the general case). In fact, by the area theorem (see, for example, [11, Chapter 2, Section 4, p. 49])

$$|c|^2 \geqslant |c_1|^2 + 2|c_2|^2 \tag{5.1}$$

Consequently $|c_1c^{-1}| < 1$ and the following relations hold

$$1-\frac{|c_1|}{|c|}-\frac{1}{\varkappa}\frac{|c_2|^2}{|c|^2} > \frac{1}{2}\left(1-\frac{|c_1|^2}{|c|^2}\right)-\frac{1}{\varkappa}\frac{|c_2|^2}{|c|^2} = \frac{1}{2}\left(1-\frac{|c_1|^2}{|c|}-\frac{2}{\varkappa}\frac{|c_2|^2}{|c|^2}\right) \tag{5.2}$$

The expression on the right of the equality sign in (5.2) is positive by virtue of (5.1) and the condition $\varkappa > 1$.

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